Geometric formulation of Carnot's theorem

2001 J. Phys. A: Math. Gen. 34 1691

(http://iopscience.iop.org/0305-4470/34/8/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1691–1712 www.iop.org/Journals/ja PII: S0305-4470(01)15062-6

Geometric formulation of Carnot's theorem

Alberto Ibort[1], Manuel de León[2], Ernesto A Lacomba[3], Juan C Marrero[4], David Martín de Diego[2] and Paulo Pitanga[5]

[1] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
[2] Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, 28006 Madrid, Spain
[3] Departamento de Matemáticas, UAM-Iztapalapa, Apto Postal 55-534, 09340 Méjico, DF, Mexico
[4] Dpto de Matemática Fundamental, Facultad de Matemáticas, Universidad de La Laguna, Tenerife, Canary Islands, Spain
[5] Instituto de Fisica, Departamento de Fisica Matematica, Universidade Federal de Rio de Janeiro, Cidade Universitaria, CT Bloco A, 21944, Rio de Janeiro, RJ, Brazil

E-mail: albertoi@math.uc3m.es, mdeleon@imaff.cfmac.csic.es, lace@xanum.uam.mx, jcmarrer@ull.es, d.martin@imaff.cfmac.csic.es and pitanga@if.ufrj.br

Received 28 June 2000, in final form 2 January 2001

Abstract

A geometrical approach for one-sided constraints is given. The Riemannian metric is used in order to define convenient projectors which give the post-impulses in terms of the pre-impulses. A formulation of Carnot's theorem within this geometric framework is exhibited.

PACS numbers: 0320, 0570, 3510

1. Introduction: the problem of impulsive non-holonomic constraints

In [8, 9] we have started a geometric study of mechanical systems subjected to impulsive constraints, that is, those constraints which act instantaneously on the system, and produce jumps of momenta. A typical example is a particle or a rigid body impacting against a wall. The system could be submitted to additional non-holonomic constraints (imagine, for instance, a collision of a rolling ball), and, moreover, after the collision, some new non-holonomic constraints appearing there, could remain. This type of systems does not appear to have received sufficient attention in the literature on geometric mechanics (see [12, 20] as exceptions). However, they have been studied extensively in classical books [2, 16, 21], for instance, and more recently see [5, 10, 15, 17, 18]. For more information about recent results on rigid-body dynamics with impact see, for example [4, 19]. Not having a general enough geometric framework within which to study them, makes it difficult to understand which part of the ideas and techniques invented to describe such systems can be extended and used in more general circumstances, such as those described at the end of this paper on time-dependent impulsive non-holonomic constraints in mechanical systems.

The problem of impulsive constraints in mechanical systems is a particular instance of the more general problem of dynamical systems on manifolds having a boundary as well as a family of submanifolds. For concreteness, let M be such a manifold, with boundary ∂M, $N = N_1 \cup \cdots \cup N_r$ a family of submanifolds on M and let Γ be a vector field on M. Manifolds with a boundary will model situations where the state space of the system will be limited by external causes (a wall or a collision for particle systems, a cut-off in the momenta for Hamiltonian systems, inequality constraints for control systems, etc). The submanifolds N_k model situations where, either we want the system to remain on them, such as permanent non-holonomic constraints in Lagrangian mechanics determined by the rolling condition, etc, or constraints that the system could encounter during its evolution. Imagine, for instance a sphere that is moving without friction on a plane and in a certain region, the plane is rough and the sphere starts rolling without sliding. In this case the interpretation of the submanifolds N_k will be that of impulsive non-holonomic constraints. The most hideous situations happen along the intersections of the boundary with the submanifolds N_k. It could perfectly well happen that a trajectory of the dynamical systems ends up on a point in subsets of the form, $N_1 \cap N_2 \cap \partial M$. These situations are poorly understood and we will not dwell on them here.

We can reproduce the above discussion first in Lagrangian dynamics. The practice and experience with Lagrangian mechanics has produced a clear cut separation between holonomic and non-holonomic constraints and impulsive and non-impulsive constraints. If Q denotes the configuration space of a Lagrangian system, the first ones correspond to fixing a submanifold $N \subset TQ$ of the form $N = TP$ for some submanifold $P \subset Q$, whereas the non-holonomic constraints will be determined by general submanifolds in TQ, for instance, linear non-holonomic constraints are defined by a distribution $D \subset TQ$. On the other hand, impulsive holonomic constraints are modelled by a boundary in the manifold Q. This boundary $\partial Q \subset Q$, induces a boundary on TQ which is the restriction $T_{\partial Q}Q$ of the bundle TQ to ∂Q. Impulsive non-holonomic constraints will be correspondingly modelled by more general boundaries on TQ (i.e. given submanifolds in TQ). Needless to say all of these notions can be (and in many applications are in fact) time-dependent. Think, for instance, of a chart that is moving along a ramp with a ball jumping on top of it. The difference between non-holonomic constraints as submanifolds of TQ and impulsive non-holonomic constraints is that ordinary non-holonomic constraints are assumed to be permanent, i.e. the integral curves of the system must always lie on the submanifold N, whereas, the impulsive non-holonomic constraints act only when (and possibly after) the system hits the submanifold N.

The general problem of modelling a dynamical system on a manifold M with boundary ∂M poses the problem of being more precise about the meaning of dynamics. Because the initial-value problem for a vector field Γ on such space does not have solution in general, we cannot simply associate evolution with such an object. Further information is needed. Obviously the problem for defining evolution (even locally) occurs in $\partial M \cup N_1 \cup \cdots \cup N_r$, because elsewhere in M the Cauchy problem always has a local solution. We shall assume for the moment that the boundary ∂M is a smooth manifold of codimension one of M. The restriction of the vector field to $T_{\partial M}M$ is not in general tangent to ∂M. If this were the case there will be no problem at all and the dynamics will be well defined by a flow on the manifold M that will restrict to the invariant submanifold ∂M. In contrast, hitting the boundary means that the system is interacting with some 'external world' that we do not want, or we cannot, describe in detail. However, we can model what the consequences of this interaction will be. Changes in the state of the system are expected. It is plausible to assume that these changes are of an instantaneous nature. An instantaneous change, like the one happening with an ideal collision in a wall, will be modelled by a map $\Delta\colon \partial M \to \partial M$, that will tell us where in state space the system will emerge once it hits the boundary. If the system after hitting the boundary is going to remain on the boundary it will be modelled by a one-parameter family of diffeomorphisms on the boundary Δ_t that will describe how the state on the boundary evolves as the time passes. This situation will correspond, for instance, to when the system gets trapped in the boundary. Such a situation will occur, for instance, in a completely inelastic collision along the normal component to a wall. The geometrical model for such an example will consist of a vector field $\Gamma_{\partial M}$ on ∂M and the collision with the boundary will be described by projecting the vector field $\Gamma|_{\partial M}$ to $\Gamma_{\partial M}$. Note that if the system starts in a state on the boundary the dynamical evolution is given directly by $\Gamma_{\partial M}$. The problem of non-holonomic constraints in Lagrangian dynamics is exactly this. We shall think of the constraint submanifold N as the topological boundary of the manifold $TQ - N$. Thus the dynamical system starting on a point in N is trapped by the boundary. Specifying the dynamics consists in projecting the Euler–Lagrange vector field to N.

We shall not address the general problem of dynamical systems on manifolds with boundary and submanifolds here but, as in previous papers, we concentrate on mechanical systems in the Lagrangian formalism. This approach will help us in comparing the scope of geometrical methods with respect to classical treatments and to analyse particular examples of interest. The mechanical systems considered here are described by a Lagrangian function which is the kinetic energy of a Riemannian metric g on the configuration manifold Q minus the potential energy V. The metric provided by the kinetic energy allows us to define projectors whenever necessary, thus providing an effective way of defining the equations of motion in the boundaries (or after the impulses).

The presence of boundaries also changes the notion of conserved quantities. Thus, a function F on a manifold with a boundary which is invariant with respect to a vector field Γ is not necessarily a constant of the motion. In fact, if the dynamics along the boundary, the above-mentioned Δ map, for instance, is such that $\Delta^* F \neq F$, the function F will not be constant along the trajectories of the system. In particular, in the Hamiltonian and/or Lagrangian system, the energy of the system is not necessarily a constant of the motion. In the Lagrangian formalism the kind of dynamics on the boundary which is assumed to occur preserves the configuration manifold, i.e. if we have a manifold Q with boundary ∂Q, then the map $\Delta\colon T_{\partial Q}Q \to T_{\partial Q}Q$ must be such that $\tau_Q \circ \Delta = \tau_Q$, in other words, after hitting the boundary, the position of the system will not change instantaneously. It is obvious that continuous functions on Q such as the potential energy will remain invariant under such kinds of transformations.

A completely elastic collision against the wall will be described by the map $\Delta(q, v^{\|}+v^{\perp}) = (q, v^{\|} - v^{\perp})$, for all $q \in \partial Q$, $v = v^{\|} + v^{\perp}$ in $T_q Q = T_q(\partial Q) \oplus T_q(\partial Q)^{\perp}$. In this case, the kinetic energy will also be preserved, and hence the total energy. However, in general this does not have to be so. We will analyse the general situation for the change of the kinetic energy under several impulsive constraints (or boundaries on TQ). In the simplest case of holonomic boundaries this result is known as Carnot's theorem (see, for instance, [15, 18]). In this paper we will extend Carnot's theorem to non-holonomic impulsive constraints, and to the time-dependent holonomic and non-holonomic instantaneous constraints. Thus we will show that Carnot's theorem is a geometric consequence of the application of a convenient orthogonal projection of velocities on the manifold, which is compatible with the constraints.

It should be remarked that the approach in this paper differs from that in [8, 9], where we worked on the phase space, while we are now working on the configuration manifold itself. Indeed, the present approach permits the explanation of Carnot's theorem according to classical textbooks and, in addition, it simplifies the computations of the projectors.

The paper is organized as follows. In section 2 we briefly recall the basic ingredients of non-holonomic and impulsive constraints. Section 3 is devoted to studying the case of time-independent holonomic one-sided constraints and to derive different versions of Carnot's theorem (theorems 3.2, 3.6, 3.9 and 3.11). The time-dependent case is discussed in section 4. Several examples are given in the paper in order to illustrate the results.

2. Classical mechanical systems

2.1. Mechanical systems

Let Q be a differentiable manifold of dimension n, with local coordinates (q^i). We consider fibred coordinates $(q^i, \dot{q}^i)$ on the tangent bundle TQ such that the canonical projection $\tau_Q : TQ \longrightarrow Q$ reads as $\tau_Q(q^i, \dot{q}^i) = (q^i)$.

A classical mechanical system is determined by a Lagrangian function

$$L(v) = \tfrac{1}{2}g(v, v) - (V \circ \tau_Q)(v) \qquad v \in TQ \tag{1}$$

where g is a Riemannian metric on Q, and V is a function on the configuration space Q (the potential).

The Euler–Lagrange equations for this Lagrangian L,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad 1 \leqslant i \leqslant n$$

take the form

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = -g^{ij}\frac{\partial V}{\partial q^j} \tag{2}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection ∇ determined by g. In terms of the Levi-Civita connection we can rewrite equation (2) in a more compact and intrinsic way. Indeed, a curve $c : I \longrightarrow Q$ of class C^2, $c(t) = (q^1(t), \ldots, q^n(t))$, is a solution for the Lagrangian system if and only if

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\mathrm{grad}\, V(c(t))$$

where the gradient is considered with respect to g (see [1]).

2.2. Non-holonomic mechanical systems

Suppose in addition that the system is subjected to non-holonomic constraints $\phi^a : TQ \longrightarrow \mathbb{R}$, $1 \leqslant a \leqslant m$, that define a submanifold $N = \{v \in TQ | \phi^a(v) = 0, a = 1, \ldots, m\}$ of TQ. In this case, not all the velocities are admissible but only those compatible with the constraints.

The non-holonomic constraints usually found in mechanics are linear in the velocities, that is, they are of the form

$$\phi^a(q, \dot{q}) = \mu^a_i(q)\dot{q}^i \qquad 1 \leqslant a \leqslant m.$$

As is well known the equations of motion for the non-holonomic problem can be derived from d'Alembert's principle:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right]\delta q^i = 0 \tag{3}$$

where δq^i denotes the virtual displacements, verifying

$$\mu^a_i \delta q^i = 0. \tag{4}$$

(We will assume that the system is not subjected to non-conservative forces.) In terms of Lagrange multipliers we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \mu_i^a.$$

From a more geometrical point of view the linear constraints are determined by prescribing a distribution $\mathcal{D}$ on Q of dimension $n - m$ such that the annihilator of $\mathcal{D}$ is locally given by

$$\mathcal{D}^o = \langle \mu^a = \mu_i^a dq^i; 1 \leqslant a \leqslant m \rangle$$

with μ^a a family of independent 1-forms on Q. In this manner, the solutions of the non-holonomic Lagrangian system satisfy

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\operatorname{grad} V(c(t)) + \lambda(\dot{c}(t)) \qquad \dot{c}(t) \in \mathcal{D}_{c(t)} \tag{5}$$

where λ is a section of $\mathcal{D}^\perp$ along c and, $\mathcal{D}^\perp$ denotes the orthogonal complement of $\mathcal{D}$ with respect to the metric g.

Since g is a Riemannian metric, the $m \times m$ matrix $(C^{ab}) = (\mu_i^a g^{ij} \mu_j^b)$ is symmetric and regular. Therefore, we can obtain the Lagrange multipliers as

$$\lambda(q^i(t), \dot{q}^i(t)) = C_{ab}\left(\left(-\Gamma^i_{jk}\dot{q}^j\dot{q}^k - g^{ij}\frac{\partial V}{\partial q^j}\right)\mu_i^a + \dot{q}^i\dot{q}^j\frac{\partial \mu_i^a}{\partial q^j}\right) Z^b$$

where (C_{ab}) is the inverse matrix of (C^{ab}) and, the vector field Z^a is defined by

$$g(Z^a, Y) = \mu^a(Y) \qquad \text{for any vector field } Y \quad 1 \leqslant a \leqslant m$$

that is, Z^a is the gradient of the 1-form μ^a. Thus, $\mathcal{D}^\perp = \langle Z^a \rangle, 1 \leqslant a \leqslant m$. In local coordinates, we have

$$Z^a = g^{ij}\mu_i^a \frac{\partial}{\partial q^j}.$$

By using the metric g and the distribution $\mathcal{D}$ we can obtain two complementary orthogonal projectors

$$\mathcal{P} : TQ \to \mathcal{D}$$
$$\mathcal{Q} : TQ \to \mathcal{D}^\perp$$

with respect to g.

A direct computation shows that the projector $\mathcal{Q}$ is given by

$$\mathcal{Q} = C_{ab} Z^a \otimes \mu^b.$$

Using these projectors we have that the equations of motion are the following. A curve $c(t)$ is a motion for the non-holonomic system if it satisfies the constraints, say, $\phi^a(\dot{c}(t)) = 0$, for all a, and, in addition, the 'projected equation of motion'

$$\mathcal{P}(\nabla_{\dot{c}(t)}\, \dot{c}(t)) = -\mathcal{P}(\operatorname{grad} V(c(t)))$$

is fulfilled. But these conditions are equivalent to

$$\dot{c}(t) \in \mathcal{D}_{c(t)} \qquad \bar{\nabla}_{\dot{c}(t)}\dot{c}(t) = -\mathcal{P}(\operatorname{grad} V(c(t)))$$

where $\bar{\nabla}$ is the modified linear connection defined by

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \mathcal{Q})(Y)$$

for all vector fields X and Y on Q (see [7, 13] for more details).

Remark 2.1. Another possible approach to the Lagrangian formulation of the dynamics of systems with non-holonomic constraints is Hamel's method of quasi-coordinates (or pseudo-coordinates). It was revisited by Koiller [11] (see also [3]) in terms of principal bundles (see also [14]), and this analysis gave a new impulse to the study of non-holonomic systems. A recent collection of papers on the subject is [6].

2.3. Impulsive constraints

Consider a Lagrangian system subjected to m permanent non-holonomic constraints $\phi^a = \mu_i^a \dot{q}^i$, $1 \leqslant a \leqslant m$. Suppose that at time t_0 new constraints $\Psi^A = \nu_i^A(q)\dot{q}^i$, $0 \leqslant A \leqslant l$ are instantaneously imposed on the system. (Without loss of generality, in the following, we will assume that these new constraints and the permanent ones are independent.) Therefore, the virtual displacements must verify (4) and in addition the equations

$$\nu_i^A \delta q^i = 0.$$

Now, if $c(t) = (q^1(t), q^2(t), \ldots, q^n(t))$ is a trajectory of the system (subjected to permanent constraints ϕ^a and impulsive constraints Ψ^A at time t_0), then for all intervals $[t_1, t_2]$ in its domain, we must have

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q^i} \delta q^i(t) + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i(t) \right] = 0$$

where $\delta q^i(t)$ is any variation with fixed points at t_1 and t_2 and, moreover, $\mu_i^a \delta q^i(t) = 0$, $1 \leqslant a \leqslant m$, and $\nu_i^A \delta q^i(t_0) = 0$, $0 \leqslant A \leqslant l$.

In particular, with $t_1 = t_0 - \epsilon$ and $t_2 = t_0 + \epsilon$, it follows that

$$p_i(t_0 + \epsilon)\delta q^i(t_0 + \epsilon) - p_i(t_0 - \epsilon)\delta q^i(t_0 - \epsilon) = \int_{t_0-\epsilon}^{t_0+\epsilon} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) \mathrm{d}t$$

$$= \int_{t_0-\epsilon}^{t_0+\epsilon} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right] \delta q^i \, \mathrm{d}t.$$

The last term is zero using d'Alembert's principle for any admissible variations. Now, taking limits as ϵ tends to zero, we have that

$$\Delta p_i \delta q^i = (p_i(t_0^+) - p_i(t_0^-))\delta q^i(t_0) = 0$$

or, by using Lagrange multipliers,

$$\Delta p_i = \bar{\lambda}_a \mu_i^a + \bar{\mu}_A \nu_i^A.$$

But $p_i = g_{ij}\dot{q}^j$, so that we have

$$\Delta \dot{q}^i = \dot{q}^i(t_0^+) - \dot{q}^i(t_0^-) = \bar{\lambda}_a g^{ij} \mu_j^a + \bar{\mu}_A g^{ij} \nu_j^A \tag{6}$$

where $\dot{q}^i(t_0^+)$ and $\dot{q}^i(t_0^-)$ are the velocities before and after the impulse, respectively (see [15, 18]).

Remark 2.2. The authors wish to thank an anonymous referee for pointing out to us the justification of equation (6).

In the next sections we will apply the above formalism to several particular cases.

3. Time-independent holonomic or non-holonomic instantaneous constraints

3.1. Mechanical systems subjected to a holonomic one-sided constraint. Carnot's theorem

Consider a Lagrangian system subjected to m permanent non-holonomic linear constraints $\phi^a = \mu^a_i(q)\dot{q}^i$ and in addition to a holonomic one-sided constraint $\Psi(q) \geqslant 0$ (this would be the case of collision with a fixed wall, for instance). The inequality $\Psi(q) \geqslant 0$ determines a closed subset of Q whose boundary N is a $(n-1)$-dimensional submanifold of Q (see [9]).

From (5) and (6), the equations of motion of the non-holonomic system subjected to this impulsive constraint are the following:

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\operatorname{grad} V(c(t)) + \lambda(\dot{c}(t)) \quad \text{and} \quad \dot{c}(t) \in \mathcal{D}(c(t)) \qquad \text{if} \quad \Psi(c(t)) > 0$$

$$\Delta\dot{c}(t) = \dot{c}(t^+) - \dot{c}(t^-) \in \mathcal{D}^\perp(c(t)) + T^\perp_{c(t)}N \qquad \text{if} \quad \Psi(c(t)) = 0.$$

Therefore, if $c(t) = (q^i(t))$ we have

$$\dot{q}^i(t^+) - \dot{q}^i(t^-) = \bar{\lambda}_a g^{ij}\mu^a_j + \bar{\mu} g^{ij}\frac{\partial\Psi}{\partial q^j}.$$

Since $\dot{c}(t^+)$ and $\dot{c}(t^-) \in \mathcal{D}(c(t))$ we deduce that

$$\Delta\dot{c}(t) \in \mathcal{P}(T^\perp_{c(t)}N)$$

or, equivalently,

$$\Delta\dot{c}(t) = \left(\bar{\mu}\ \operatorname{grad}\Psi - \bar{\mu}C_{ab}\left[\mu^a(\operatorname{grad}\Psi)\right]Z^b\right)_{|c(t)}.$$

From this last equation we observe that, in order to know the post-velocities (after the impulse) from the pre-velocities (before the impulse), it is only necessary to determine the remaining Lagrange multiplier $\bar{\mu}$.

We are assuming that during the impact the unique impulsive force acting is due to the restitution of the wall determined by $\Psi(q) \geqslant 0$. Suppose that the restitution coefficient is $\alpha \in [0, 1]$.

The submanifold N and the projector $\mathcal{P}$ determine two new complementary orthogonal projectors along the points of N:

$$\tilde{\mathcal{Q}} : TQ \rightarrow \mathcal{P}(T^\perp N)$$

$$\tilde{\mathcal{P}} : TQ \rightarrow (\mathcal{P}(T^\perp N))^\perp$$

defined locally by

$$\tilde{\mathcal{Q}} = \mathcal{C}^{-1}\,\mathcal{P}(\operatorname{grad}\Psi) \otimes \beta \tag{7}$$

where $\beta(X) = g(\mathcal{P}(\operatorname{grad}\Psi), X)$, for all X, $\mathcal{C} = g(\mathcal{P}(\operatorname{grad}\Psi), \mathcal{P}(\operatorname{grad}\Psi))$ and $\tilde{\mathcal{P}} = \mathrm{id} - \tilde{\mathcal{Q}}$. Note that $\mathcal{C} \neq 0$ since we are assuming that the 1-forms μ^a and $\mathrm{d}\Psi$ are independent along N.

We assume that the normal components of the velocities before and after the impact are related by the formula

$$\dot{c}(t^+)^\perp = -\alpha\,\dot{c}(t^-)^\perp \tag{8}$$

where α is the restitution coefficient of the wall. In other words, we have

$$\mathrm{d}\Psi(\dot{c}(t^+)) = -\alpha\,\mathrm{d}\Psi\,(\dot{c}(t^-)). \tag{9}$$

A direct computation, using (7) and the fact that $\dot{c}(t^-)$ and $\dot{c}(t^+)$ belong to $\mathcal{D}(c(t))$, proves that (9) is equivalent to

$$\tilde{\mathcal{Q}}(\dot{c}(t^+)) = -\alpha\tilde{\mathcal{Q}}(\dot{c}(t^-)).$$

Thus, since $\tilde{\mathcal{P}}(\dot{c}(t^+)) = \tilde{\mathcal{P}}(\dot{c}(t^-))$ we obtain that

$$\dot{c}(t^+) = (\tilde{\mathcal{P}} - \alpha\tilde{\mathcal{Q}})(\dot{c}(t^-)). \tag{10}$$

The following lemma will be useful in the proof of Carnot's theorem.

Lemma 3.1. *Let V be a real vector space endowed with an inner product $\langle\, , \rangle$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal linear endomorphisms of V, that is, $\langle \mathcal{A}(u), \mathcal{B}(v)\rangle = 0$, for all $u, v \in V$, and that $(\mathcal{A} + \mathcal{B})(v) = v$, for any $v \in \operatorname{Im}\mathcal{A} \oplus \operatorname{Im}\mathcal{B}$. Consider the endomorphism $\mathcal{A} - \alpha\mathcal{B}$, where $\alpha \in [0, 1]$. We have*

$$\langle(\mathcal{A} - \alpha\mathcal{B})(v), (\mathcal{A} - \alpha\mathcal{B})(v)\rangle - \langle v, v\rangle = -\frac{1-\alpha}{1+\alpha}\langle \mathcal{A}(v) - \alpha\mathcal{B}(v) - v, \mathcal{A}(v) - \alpha\mathcal{B}(v) - v\rangle$$

for any $v \in \operatorname{Im}\mathcal{A} \oplus \operatorname{Im}\mathcal{B}$.

Proof. Since $v \in \operatorname{Im}\mathcal{A} \oplus \operatorname{Im}\mathcal{B}$, we have that

$$\begin{aligned}\langle(\mathcal{A} - \alpha\mathcal{B})(v), (\mathcal{A} - \alpha\mathcal{B})(v)\rangle - \langle v, v\rangle &= \langle(\mathcal{A} - \alpha\mathcal{B})(v), (\mathcal{A} - \alpha\mathcal{B})(v)\rangle \\ &\quad -\langle \mathcal{A}(v) + \mathcal{B}(v), \mathcal{A}(v) + \mathcal{B}(v)\rangle \\ &= (\alpha^2 - 1)\langle \mathcal{B}(v), v\rangle.\end{aligned}$$

Moreover,

$$\langle \mathcal{A}(v) - \alpha\mathcal{B}(v) - v, \mathcal{A}(v) - \alpha\mathcal{B}(v) - v\rangle = (1+\alpha)^2\langle \mathcal{B}(v), v\rangle.$$

The result now follows from both equalities. □

Theorem 3.2 (Standard Carnot's theorem [15]). *If T_1 is the kinetic energy after the impulse, T_0 the kinetic energy before the impulse and T_l the kinetic energy of the loss in velocity, then we have*

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha}T_l$$

where α is the restitution coefficient of the wall.

Proof. Carnot's theorem follows from (10) and lemma 3.1. Indeed, observe that

$$\begin{aligned}T_1 &= \tfrac{1}{2}g(\dot{c}(t^+), \dot{c}(t^+)) \\ T_0 &= \tfrac{1}{2}g(\dot{c}(t^-), \dot{c}(t^-)) \\ T_l &= \tfrac{1}{2}g(\dot{c}(t^+) - \dot{c}(t^-), \dot{c}(t^+) - \dot{c}(t^-)).\end{aligned}$$

□

Remark 3.3. Note that, in order to obtain theorem 3.2, we have applied lemma 3.1 in a particular case, when the two orthogonal endomorphisms fill up the whole tangent bundle.

3.2. Mechanical systems subjected to instantaneous non-holonomic constraints

In this situation, we will assume that, apart of the holonomic one-sided constraint $\Psi(q) \geqslant 0$, non-holonomic linear constraints on the velocities are imposed instantaneously. Suppose that these new constraints are given by

$$\psi^A(q^i, \dot{q}^i) = \nu_i^A(q)\dot{q}^i \qquad 1 \leqslant A \leqslant l.$$

At any point q such that $\Psi(q) = 0$, we consider the vector space $F(q)$ annihilated by

$$\langle (\nu^A = \nu_i^A \, \mathrm{d}q^i)_{|q} \rangle \qquad 1 \leqslant A \leqslant l.$$

In this case, the equations of motion of the mechanical system are:

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\operatorname{grad} V(c(t)) + \lambda(\dot{c}(t)) \quad \text{and} \quad \dot{c}(t) \in \mathcal{D}(c(t)) \qquad \text{if} \quad \Psi(c(t)) > 0$$

$$\Delta\dot{c}(t) = \dot{c}(t^+) - \dot{c}(t^-) \in \mathcal{D}^\perp(c(t)) + T^\perp_{c(t)}N + F^\perp(c(t)) \qquad \text{if} \quad \Psi(c(t)) = 0.$$

Thus, if $c(t) = (q^i(t))$ we have

$$\dot{q}^i(t^+) - \dot{q}^i(t^-) = \bar{\lambda}_a g^{ij}\mu_j^a + \bar{\mu} g^{ij}\frac{\partial \Psi}{\partial q^j} + \bar{\mu}_A g^{ij}\nu_j^A.$$

Using the same technique as above, we obtain that

$$\Delta\dot{c}(t) \in \mathcal{P}(T^\perp_{c(t)}N + F^\perp(c(t)))$$

or, equivalently,

$$\Delta\dot{c}(t) = \left(\bar{\mu} \operatorname{grad}\Psi - \bar{\mu} C_{ab}\mu^a(\operatorname{grad}\Psi)Z^b + \bar{\mu}_A Y^A - \bar{\mu}_A C_{ab}\mu^a(Y^A)Z^b\right)_{|c(t)}$$

where Y^A, $1 \leqslant A \leqslant l$ are the vector fields along N defined by $g(Y^A, X) = \nu^A(X)$, for any X.

In order to determine the Lagrange multipliers $\bar{\mu}$ and $\bar{\mu}_A$ it is necessary to use additional physical conditions. In our case, we will suppose that the restitution coefficient of the wall is $\alpha \in [0, 1]$, and that the instantaneous non-holonomic constraints remain after the impulse.

Define the following complementary projectors along the points of N:

$$\mathbb{Q} : TQ \longrightarrow \mathcal{P}(T^\perp N + F^\perp)$$

$$\mathbb{P} : TQ \longrightarrow (\mathcal{P}(T^\perp N + F^\perp))^\perp.$$

Since the impulsive and permanent constraints are independent, we deduce that the 1-forms $\{\mu^A, \mathrm{d}\Psi, \nu^B\}$ are independent along the points of N. Thus, if $Y^0 = \operatorname{grad}\Psi$ then the matrix $\left((\mathcal{C}^{AB} = g(\mathcal{P}(Y^A), \mathcal{P}(Y^B)))\right)_{0 \leqslant A \leqslant B \leqslant l}$ is regular. Now, a direct computation shows that

$$\mathbb{Q} = \sum_{0 \leqslant A, B \leqslant l} \mathcal{C}_{AB}\, \mathcal{P}(Y^A) \otimes \beta^B$$

where $\beta^B(X) = g(\mathcal{P}(Y^B), X)$, for all X, and $\mathcal{C}_{AB}$ denotes the entries of the inverse matrix of the matrix $(\mathcal{C}^{AB})$.

Proposition 3.4. *Assuming the above conditions, we obtain that*

$$\dot{c}(t^+) = (\mathbb{P} - \alpha\tilde{\mathcal{T}})\left(\dot{c}(t^-)\right) \tag{11}$$

where

$$\tilde{\mathcal{T}} = \sum_{0 \leqslant A \leqslant l} \mathcal{C}_{A0}\, \mathcal{P}(Y^A) \otimes \beta^0.$$

Proof. From the construction of the projectors $\mathbb{P}$ and $\mathbb{Q}$ we have

$$\mathbb{P}(\dot{c}(t^+)) = \mathbb{P}(\dot{c}(t^-))$$

and, consequently, we deduce that

$$\begin{aligned}\dot{c}(t^+) &= \mathbb{P}(\dot{c}(t^-)) + \mathbb{Q}(\dot{c}(t^+)) \\ &= \mathbb{P}(\dot{c}(t^-)) + \left[\sum_{0\leqslant A,B\leqslant l} \mathcal{C}_{AB}\, \mathcal{P}(Y^A) \otimes \beta^B\right](\dot{c}(t^+)) \\ &= \mathbb{P}(\dot{c}(t^-)) + \sum_{0\leqslant A,B\leqslant l} \mathcal{C}_{AB}\left[\beta^B(\dot{c}(t^+))\right]\mathcal{P}(Y^A).\end{aligned}$$

Since the non-holonomic instantaneous constraints are preserved after the impact we have

$$\nu^A(\dot{c}(t^+)) = 0 \qquad 1 \leqslant A \leqslant l.$$

In addition, $\beta^A - \nu^A \in \mathcal{D}^o$, $1 \leqslant A \leqslant l$, since $(\beta^A - \nu^A)(X) = g(\mathcal{P}(Y^A), X) - \nu^A(X) = g(Y^A, X) - \nu^A(X) = 0$ if $X \in \mathcal{D}$. Therefore, we obtain

$$\beta^A(\dot{c}(t^+)) = 0 \qquad 1 \leqslant A \leqslant l.$$

We are assuming that the normal components of the velocities before and after the impulse are related by (8), which is in turn equivalent to

$$\tilde{\mathcal{Q}}(\dot{c}(t^+)) = -\alpha\tilde{\mathcal{Q}}(\dot{c}(t^-))$$

and hence

$$\beta^0(\dot{c}(t^+)) = -\alpha\beta^0(\dot{c}(t^-)).$$

(Note that β^0 is the 1-form corresponding to $\mathcal{P}(Y^0)$, according to the definition of the β^As.) Finally, we have

$$\dot{c}(t^+) = \mathbb{P}(\dot{c}(t^-)) - \alpha \sum_{0\leqslant A\leqslant l} \mathcal{C}_{A0}\left[\beta^0(\dot{c}(t^-))\right]\mathcal{P}(Y^A). \qquad \square$$

Lemma 3.5. *$\mathbb{P}$ and $\tilde{\mathcal{T}}$ are orthogonal and $(\tilde{\mathcal{T}} + \mathbb{P})(X) = X$, for all $X \in \operatorname{Im}\mathbb{P} \oplus \operatorname{Im}\tilde{\mathcal{T}}$.*

Proof. The proof is straightforward from the local expressions of both projectors. $\square$

Theorem 3.6 (Carnot's theorem). *If T_1 is the kinetic energy after the impulse, T_0 the kinetic energy before the impulse and T_l the kinetic energy of the loss in velocity, and if the initial velocity verifies $\dot{c}(t^-) \in \operatorname{Im}\mathbb{P} \oplus \operatorname{im}\tilde{\mathcal{T}}$, we have*

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha}T_l$$

where α is the restitution coefficient of the wall.

Proof. The result follows from proposition 3.4 and lemmas 3.1 and 3.5. $\square$

Example 3.7. While moving in a vertical plane xOy a circular disc of radius R and mass m hits a rough wall determined by the axis $0x$. Assuming that the motion is planar, the system possesses three degrees of freedom: the coordinates x and y of the centre of the disc and θ the angle between a point P of the disc and the axis $0y$.

The system is described by the Lagrangian function

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2\dot{\theta}^2\right)$$

where mk^2 denotes the moment of inertia of the disc. There are no permanent non-holonomic constraints but, in addition, apart of the holonomic one-sided constraint $\Psi = y - R$ we have an impulsive constraint along the line $y = R$:

$$\psi^1 = \dot{x} - R\dot{\theta}.$$

Following the notation introduced above, we have

$$Y^0 = \frac{1}{m}\frac{\partial}{\partial y} \qquad Y^1 = \frac{1}{m}\frac{\partial}{\partial x} - \frac{R}{mk^2}\frac{\partial}{\partial \theta}$$

and

$$\begin{pmatrix} \mathcal{C}^{00} & \mathcal{C}^{01} \\ \mathcal{C}^{10} & \mathcal{C}^{11} \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ 0 & \dfrac{k^2 + R^2}{mk^2} \end{pmatrix}.$$

Therefore, we obtain that

$$\mathbb{Q} = \left(\frac{1}{m}\frac{\partial}{\partial y}, \frac{1}{m}\frac{\partial}{\partial x} - \frac{R}{mk^2}\frac{\partial}{\partial \theta}\right)\begin{pmatrix} m & 0 \\ 0 & \dfrac{mk^2}{R^2 + k^2} \end{pmatrix}\begin{pmatrix} \mathrm{d}y \\ \mathrm{d}x - R\,\mathrm{d}\theta \end{pmatrix}$$

$$\tilde{\mathcal{T}} = \frac{\partial}{\partial y} \otimes \mathrm{d}y.$$

Consequently, we deduce that

$$(\dot{c}_x(t^+), \dot{c}_y(t^+), \dot{c}_\theta(t^+)) = \left(\mathbb{P} - \alpha\tilde{\mathcal{T}}\right)(\dot{c}_x(t^-), \dot{c}_y(t^-), \dot{c}_\theta(t^-))$$

and

$$\dot{c}_x(t^+) = \frac{R^2\dot{c}_x(t^-) + Rk^2\dot{c}_\theta(t^-)}{R^2 + k^2}$$

$$\dot{c}_y(t^+) = -\alpha\dot{c}_y(t^-)$$

$$\dot{c}_\theta(t^+) = \frac{R\dot{c}_x(t^-) + k^2\dot{c}_\theta(t^-)}{R^2 + k^2}$$

where we have used the obvious notation for the velocity components. For pre-impact velocities we have

$$\dot{c}(t^-) \in \operatorname{Im}\mathbb{P} \oplus \operatorname{Im}\tilde{\mathcal{T}} = \left\langle R\frac{\partial}{\partial x} + \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y}\right\rangle$$

and then

$$T(\dot{c}(t^+)) - T(\dot{c}(t^-)) = -\frac{1-\alpha}{1+\alpha}T(\dot{c}(t^+) - \dot{c}(t^-)).$$

Note that theorem 3.6 only holds for initial velocities in the Whitney sum of the images of the projectors $\mathbb{P}$ and $\tilde{\mathcal{T}}$. For an extension to arbitrary initial velocities we will need a generalization of lemma 3.1.

Lemma 3.8. *Let V be a real vector space endowed with an inner product $\langle\, ,\rangle$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal endomorphisms of V, that is, $\langle \mathcal{A}(u), \mathcal{B}(v)\rangle = 0$, for all $u, v \in V$. Consider the linear endomorphism $\mathcal{A} - \alpha\mathcal{B}$, where $\alpha \in [0, 1]$. Suppose that $\boldsymbol{C} = \mathrm{id} - \mathcal{A} - \mathcal{B}$ is orthogonal to $\mathcal{A}$, then*

$$\langle(\mathcal{A} - \alpha\mathcal{B})(v), (\mathcal{A} - \alpha\mathcal{B})(v)\rangle - \langle v, v\rangle = -\frac{1-\alpha}{1+\alpha}\langle \mathcal{A}(v) - \alpha\mathcal{B}(v) - v, \mathcal{A}(v) - \alpha\mathcal{B}(v) - v\rangle$$
$$-\frac{2\alpha}{1+\alpha}\langle \boldsymbol{C}(v), \boldsymbol{C}(v)\rangle - 2\alpha\langle \boldsymbol{C}(v), \mathcal{B}(v)\rangle$$

for all $v \in V$.

Proof. For any $v \in V$ we can put $v = \mathcal{A}v + \mathcal{B}v + \boldsymbol{C}v$. Then

$$\begin{aligned}\langle(\mathcal{A} - \alpha\mathcal{B})(v), (\mathcal{A} - \alpha\mathcal{B})(v)\rangle - \langle v, v\rangle &= \langle \mathcal{A}(v), \mathcal{A}(v)\rangle + \alpha^2\langle \mathcal{B}(v), \mathcal{B}(v)\rangle \\ &\quad -\langle \mathcal{A}(v) + \mathcal{B}(v) + \boldsymbol{C}(v), \mathcal{A}(v) + \mathcal{B}(v) + \boldsymbol{C}(v)\rangle \\ &= (\alpha^2 - 1)\langle \mathcal{B}(v), \mathcal{B}(v)\rangle - \langle \boldsymbol{C}(v), \boldsymbol{C}(v)\rangle - 2\langle \mathcal{B}(v), \boldsymbol{C}(v)\rangle.\end{aligned}$$

Moreover,

$$\begin{aligned}&\langle \mathcal{A}(v) - \alpha\mathcal{B}(v) - v, \mathcal{A}(v) - \alpha\mathcal{B}(v) - v\rangle \\ &\quad = (1+\alpha)^2\langle \mathcal{B}(v), \mathcal{B}(v)\rangle + 2(\alpha+1)\langle \mathcal{B}(v), C(v)\rangle + \langle C(v), C(v)\rangle\end{aligned}$$

and the result follows. □

Theorem 3.9 (Extended Carnot's theorem). *If T_1 is the kinetic energy after the impulse, T_0 the kinetic energy before the impulse, T_l the kinetic energy of the loss in velocity and T_C is the kinetic energy of the velocity $\boldsymbol{C}(\dot{c}(t^-)) = \dot{c}(t^-) - \mathbb{P}(\dot{c}(t^-)) - \tilde{\mathcal{T}}(\dot{c}(t^-))$, then*

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha}T_l - \frac{2\alpha}{\alpha+1}T_C - \alpha g\big(\tilde{\mathcal{T}}(\dot{c}(t^-)), \boldsymbol{C}(\dot{c}(t^-))\big)$$

where α is the restitution coefficient of the wall.

Proof. It follows from lemma 3.8 and proposition 3.4. □

Note that, in general, $\tilde{\mathcal{T}}$ and $\boldsymbol{C} = \mathrm{id} - \mathbb{P} - \tilde{\mathcal{T}}$ are not orthogonal. Next, we will discuss the case when both projectors are orthogonal.

Lemma 3.10. *The following statements are equivalent:*

(a) $\tilde{\mathcal{T}}$ and $\boldsymbol{C}$ are orthogonal;
(b) $\tilde{\mathcal{T}} = \tilde{\mathcal{Q}}$;
(c) $\mathcal{P}(Y^0) \in F$.

Proof.

(a) $\Leftrightarrow$ (b)

Since $\mathbb{P}$ and $\tilde{\mathcal{T}}$ are orthogonal, then $\tilde{\mathcal{T}}$ and $\boldsymbol{C}$ are orthogonal iff

$$g(\tilde{\mathcal{T}}(u), v) = g(\tilde{\mathcal{T}}(u), \tilde{\mathcal{T}}(v))$$

for all $u, v \in T_N Q$. This last condition is equivalent to

$$\sum_{1 \leqslant A \leqslant l} \mathcal{C}_{A0} \mathcal{P}(Y^A) = 0$$

which implies $\mathcal{C}_{A0} = 0$, $1 \leqslant A \leqslant l$.

Now, using that $(\mathcal{C}^{AB})_{0 \leqslant A, B \leqslant l}$ is symmetric we deduce that $\mathcal{C}^{A0} = 0$, $1 \leqslant A \leqslant l$ and that

$$\mathcal{C}_{00} = \frac{1}{g(\mathcal{P}(\mathrm{grad}\Psi), \mathcal{P}(\mathrm{grad}\Psi))}.$$

Thus, $\tilde{\mathcal{T}} = \tilde{\mathcal{Q}}$.

Conversely, if $\tilde{\mathcal{T}} = \tilde{\mathcal{Q}}$ then

$$\sum_{1 \leqslant A \leqslant l} \mathcal{C}_{A0} \mathcal{P}(Y^A) = 0$$

and therefore $\tilde{\mathcal{T}}$ and $\boldsymbol{C}$ are orthogonal.

(a) $\Leftrightarrow$ (c)

Since

$$\mathcal{C}^{A0} = g(\mathcal{P}(Y^A), \mathcal{P}(Y^0)) = g(Y^A, \mathcal{P}(Y^0)) = \nu^A(\mathcal{P}(Y^0))$$

for $1 \leqslant A \leqslant l$, we conclude that $\tilde{\mathcal{T}}$ and $\boldsymbol{C}$ are orthogonal if and only if $\mathcal{P}(Y^0) \in F$. $\square$

Theorem 3.11. *If $\mathcal{P}(Y^0) \in F$ then*

$$\dot{c}(t^+) = (\mathbb{P} - \alpha \tilde{\mathcal{Q}})(\dot{c}(t^-))$$

and

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha} T_l - \frac{2\alpha}{\alpha+1} T_C$$

where α is the restitution coefficient of the wall.

Proof. It follows from theorem 3.9 and lemma 3.10. $\square$

Example 3.12. Following example 3.7, suppose now that $\dot{c}(t^-) \notin \mathrm{Im}\,\mathbb{P} \oplus \mathrm{Im}\,\tilde{\mathcal{T}}$. This condition is equivalent to $\dot{c}_x(t^-) \neq R\dot{c}_\theta(t^-)$. Observe that in this case $\tilde{\mathcal{T}} = \tilde{\mathcal{Q}}$. The projection of the initial velocity by $\boldsymbol{C} = \mathbb{Q} - \tilde{\mathcal{Q}}$ gives

$$\left(\frac{k^2}{k^2 + R^2} (\dot{c}_x(t^-) - R\dot{c}_\theta(t^-)), 0, -\frac{R}{k^2 + R^2} (\dot{c}_x(t^-) - R\dot{c}_\theta(t^-)) \right)$$

and then

$$T_C = \frac{mk^2}{2(k^2 + R^2)} (\dot{c}_x(t^-) - R\dot{c}_\theta(t^-))^2.$$

From theorem 3.11, we have that

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha} T_l - \frac{2\alpha}{\alpha+1} T_C.$$

Example 3.13. A sphere of radius r and mass 1 rolls without sliding on a horizontal plane. At the instant t_0, the sphere hits a rough wall determined by the plane $x = 0$ (see [15]).

The kinetic energy of the sphere is

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right)$$

where ω_x, ω_y and ω_z are the angular velocities given by

$$\omega_x = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi$$

$$\omega_y = \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi$$

$$\omega_z = \dot{\varphi}\cos\theta + \dot{\psi}$$

and φ, θ and ψ are the Eulerian angles.

Since the point of contact of the sphere and the plane has zero velocity, we have the following permanent constraints:

$$\phi^1 = \dot{x} - r\omega_y = 0$$

$$\phi^2 = \dot{y} + r\omega_x = 0$$

$$\phi^3 = \dot{z} = 0.$$

When the sphere hits the wall, apart of the holonomic one-sided constraint $\Psi = x$, the following constraints are instantaneously imposed:

$$\begin{aligned}\psi^1 &= \dot{y} - r\omega_z = 0\\ \psi^2 &= \omega_y = 0.\end{aligned} \tag{12}$$

We consider only $\psi^0 = \dot{\Psi} = \dot{x}$ and ψ^1 since ψ^2 is a linear combination of ϕ^1 and ψ^0.

Following the classical procedure, we introduce quasi-coordinates 'q_1', 'q_2' and 'q_3' such that '$\dot{q}_1$' $= \omega_x$, '$\dot{q}_2$' $= \omega_y$ and '$\dot{q}_3$' $= \omega_z$ (see [11, 14]). These last expressions only have a symbolic meaning where we interpret '$\mathrm{d}q_i$' and '$\frac{\partial}{\partial q_i}$', $1 \leqslant i \leqslant 3$, as adequate combinations of the differentials and partial derivatives, respectively, of the Eulerian angles. Note that $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, `\frac{\partial}{\partial q_1}', `\frac{\partial}{\partial q_2}', `\frac{\partial}{\partial q_3}'\right\}$ and $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z, `\mathrm{d}q^1', `\mathrm{d}q^2', `\mathrm{d}q^3'\}$ are dual bases.

The projector $\mathcal{Q}$, obtained from the permanent constraints, is given by

$$\mathcal{Q} = \left(\frac{\partial}{\partial x} - \frac{r}{k^2}\frac{\partial}{\partial q_2}, \frac{\partial}{\partial y} + \frac{r}{k^2}\frac{\partial}{\partial q_1}, \frac{\partial}{\partial z}\right)\begin{pmatrix} \dfrac{k^2}{k^2+r^2} & 0 & 0\\ 0 & \dfrac{k^2}{k^2+r^2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}x - r\,\mathrm{d}q_2\\ \mathrm{d}y + r\,\mathrm{d}q_1\\ \mathrm{d}z\end{pmatrix}.$$

From the constraints ψ^0 and ψ^1, which are instantaneously imposed on the system, we obtain the 1-forms:

$$\nu^0 = \mathrm{d}x$$

$$\nu^1 = \mathrm{d}y - r\,\mathrm{d}q_3$$

and the associated vector fields from the metric g are:

$$Y^0 = \frac{\partial}{\partial x}$$

$$Y^1 = \frac{\partial}{\partial y} - \frac{r}{k^2}\frac{\partial}{\partial q_3}.$$

Therefore, we have

$$\mathcal{P}(Y^0) = \frac{r^2}{k^2+r^2}\frac{\partial}{\partial x} + \frac{r}{k^2+r^2}\frac{\partial}{\partial q_2}$$

$$\mathcal{P}(Y^1) = \frac{r^2}{k^2+r^2}\frac{\partial}{\partial y} - \frac{r}{k^2+r^2}\frac{\partial}{\partial q_1} - \frac{r}{k^2}\frac{\partial}{\partial q_3}.$$

Observe that $\mathcal{P}(Y^0) \in F$, that is, $\nu^1(\mathcal{P}(Y^0)) = 0$.

Thus, the relations between the pre-impact and post-impact linear and angular velocities are obtained from the equations:

$$\Delta\dot{x} = \frac{r^2}{k^2+r^2}\bar{\mu}_0$$

$$\Delta\dot{y} = \frac{r^2}{k^2+r^2}\bar{\mu}_1$$

$$\Delta\dot{z} = 0$$

$$\Delta\omega_x = -\frac{r}{k^2+r^2}\bar{\mu}_1$$

$$\Delta\omega_y = \frac{r}{k^2+r^2}\bar{\mu}_0$$

$$\Delta\omega_z = -\frac{r}{k^2}\bar{\mu}_1$$

where $\bar{\mu}_0$ and $\bar{\mu}_1$ are undetermined Lagrange multipliers.

In order to determine the Lagrange multipliers $\bar{\mu}_0$ and $\bar{\mu}_1$ it is necessary to require additional information about the system. Indeed, assume that the restitution coefficient of the wall is α and that the instantaneous non-holonomic constraints remain after the impulse. Under these conditions, if $\dot{c}(t_0) = (\dot{x}_0, \dot{y}_0, \dot{z}_0, (\omega_x)_0, (\omega_y)_0, (\omega_z)_0)$ and $\dot{c}(t_1) = (\dot{x}_1, \dot{y}_1, \dot{z}_1, (\omega_x)_1, (\omega_y)_1, (\omega_z)_1)$ are the linear and angular velocities before and after the impact, respectively, then using that $\phi^i(\dot{c}(t_0)) = \phi^i(\dot{c}(t_1)) = 0$, we deduce that

$$\dot{x}_1 = -\alpha\dot{x}_0$$

$$\dot{y}_1 - r(\omega_z)_1 = 0.$$

The 1-forms β_0 and β_1 related to $\mathcal{P}(Y_0)$ and $\mathcal{P}(Y_1)$, respectively, by the musical isomorphism induced by the metric g are:

$$\beta_0 = \frac{r^2}{k^2+r^2}\,\mathrm{d}x + \frac{rk^2}{k^2+r^2}\,\mathrm{d}q_2$$

$$\beta_1 = \frac{r^2}{k^2+r^2}\,\mathrm{d}y - \frac{rk^2}{k^2+r^2}\,\mathrm{d}q_1 - r\,\mathrm{d}q_3.$$

Therefore, the projector $\mathbb{Q}$ is given by

$$\mathbb{Q} = A\begin{pmatrix} \dfrac{k^2+r^2}{r^2} & 0 \\ 0 & \dfrac{k^2(k^2+r^2)}{r^2(2k^2+r^2)} \end{pmatrix}\begin{pmatrix} \dfrac{r^2}{k^2+r^2}\,\mathrm{d}x + \dfrac{rk^2}{k^2+r^2}\,\mathrm{d}q_2 \\ \dfrac{r^2}{k^2+r^2}\,\mathrm{d}y - \dfrac{rk^2}{k^2+r^2}\,\mathrm{d}q_1 - r\,\mathrm{d}q_3 \end{pmatrix}$$

where

$$A = \left(\frac{r^2}{k^2+r^2}\frac{\partial}{\partial x} + \frac{r}{k^2+r^2}\frac{\partial}{\partial q_2}, \frac{r^2}{k^2+r^2}\frac{\partial}{\partial y} - \frac{r}{k^2+r^2}\frac{\partial}{\partial q_1} - \frac{r}{k^2}\frac{\partial}{\partial q_3}\right)$$

and the projector $\tilde{\mathcal{T}} = \tilde{\mathcal{Q}}$ is

$$\tilde{\mathcal{T}} = \frac{1}{k^2 + r^2}\left(r\frac{\partial}{\partial x} + \frac{\partial}{\partial q_2}\right) \otimes (r\,\mathrm{d}x + k^2\,\mathrm{d}q_2).$$

Consider now a pre-impact velocity which verifies the permanent non-holonomic constraints, that is,

$$(\dot{x}_0, \dot{y}_0, 0, -\dot{y}_0/r, \dot{x}_0/r, (\omega_z)_0).$$

Then, if we project this velocity by $\mathbb{P} - \alpha\tilde{\mathcal{T}}$ we obtain the post-impact velocities:

$$\begin{aligned}
\dot{x}_1 &= -\alpha\dot{x}_0 \\
\dot{y}_1 &= \frac{(k^2 + r^2)\dot{y}_0 + rk^2(\omega_z)_0}{2k^2 + r^2} \\
\dot{z}_1 &= 0 \\
(\omega_x)_1 &= -\frac{(k^2 + r^2)\dot{y}_0 + rk^2(\omega_z)_0}{r(2k^2 + r^2)} \\
(\omega_y)_1 &= -\alpha\frac{\dot{x}_0}{r} \\
(\omega_z)_1 &= \frac{(k^2 + r^2)\dot{y}_0 + rk^2(\omega_z)_0}{r(2k^2 + r^2)}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathrm{Im}\,\mathbb{P} &= \left\langle k^2\frac{\partial}{\partial x} - r\frac{\partial}{\partial q_2}, \frac{\partial}{\partial z}, k^2\frac{\partial}{\partial y} + r\frac{\partial}{\partial q_1}, (k^2 + r^2)\frac{\partial}{\partial y} + r\frac{\partial}{\partial q_3}\right\rangle \\
\mathrm{Im}\,\tilde{\mathcal{T}} &= \left\langle r\frac{\partial}{\partial x} + \frac{\partial}{\partial q_2}\right\rangle
\end{aligned}$$

and then

$$\left(\mathrm{Im}\,\mathbb{P} \oplus \mathrm{Im}\,\tilde{\mathcal{T}}\right) \cap \mathrm{Im}\,\mathcal{P} = \left\langle r\frac{\partial}{\partial x} + \frac{\partial}{\partial q_2}, r^2\frac{\partial}{\partial y} - r\frac{\partial}{\partial q_1} + r\frac{\partial}{\partial q_3}\right\rangle.$$

It is easy to prove that a pre-impact velocity belongs to $\left(\mathrm{Im}\,\mathbb{P} \oplus \mathrm{Im}\,\tilde{\mathcal{T}}\right) \cap \mathrm{Im}\,\mathcal{P}$ if and only if $\dot{y}_0 = r(\omega_z)_0$. Then, the relation between the kinetic energies would be

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha}T_l. \tag{13}$$

In the case $\dot{y}_0 \neq r(\omega_z)_0$, we need to add an extra term to equation (13). For that, we first find the projection by $\boldsymbol{C} = \mathrm{id} - \mathbb{P} - \tilde{\mathcal{T}} = \mathbb{Q} - \tilde{\mathcal{T}}$ of the pre-impact velocities. We denote this projection by $(\dot{x}_C, \dot{y}_C, \dot{z}_C, (\omega_x)_C, (\omega_y)_C, (\omega_z)_C)$ which is equal to

$$\left(0, \frac{k^2}{2k^2 + r^2}(\dot{y}_0 - r(\omega_z)_0), 0, -\frac{k^2}{r(2k^2 + r^2)}(\dot{y}_0 - r(\omega_z)_0), 0, -\frac{k^2 + r^2}{r(2k^2 + r^2)}(\dot{y}_0 - r(\omega_z)_0)\right).$$

Next, we compute the kinetic energy T_C for this velocity:

$$T_C = \frac{k^2(k^2 + r^2)}{2r^2(2k^2 + r^2)}(\dot{y}_0 - r(\omega_z)_0)^2$$

and then from theorem 3.11

$$T_1 - T_0 = -\frac{1-\alpha}{1+\alpha}T_l - \frac{2\alpha}{\alpha + 1}T_C.$$

Observe that for an elastic collision ($\alpha = 1$) we have

$$T_1 - T_0 = -T_C = -\frac{k^2(k^2 + r^2)}{2r^2(2k^2 + r^2)}(\dot{y}_0 - r(\omega_z)_0)^2.$$

4. Time-dependent holonomic and non-holonomic instantaneous one-sided constraints

4.1. Mechanical systems subjected to a time-dependent holonomic one-sided constraint. Carnot's theorem

Assume that we have a time-dependent holonomic one-sided constraint defined by $\Psi(t,q) \geqslant 0$. The constraint Ψ determines an $(n-1)$-dimensional unparametrized submanifold N_t of Q for any fixed time t. In fact, $N_t = \{q \in Q \ / \ \Psi_t(q) = \Psi(t,q) = 0\}$.

The equations of motion of the non-holonomic system subjected to this time-dependent impulsive constraint are

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\operatorname{grad} V(c(t)) + \lambda(\dot{c}(t)) \quad \text{and} \quad \dot{c}(t) \in \mathcal{D}(c(t)) \qquad \text{if} \quad \Psi(t,c(t)) > 0$$

$$\Delta\dot{c}(t) = \dot{c}(t^+) - \dot{c}(t^-) \in \mathcal{D}^\perp(c(t)) + T^\perp_{c(t)}N_t \qquad \text{if} \quad \Psi(t,c(t)) = 0.$$

As in subsection 3.1 we have that

$$\Delta\dot{c}(t) \in \mathcal{P}(T^\perp_{c(t)}N_t)$$

or, equivalently,

$$\Delta\dot{c}(t) = \left(\bar{\mu}\operatorname{grad}\Psi_t - \bar{\mu}C_{ab}\mu^a(\operatorname{grad}\Psi_t)Z^b\right)_{|c(t)}.$$

From this last equation we observe that, in order to know the post-velocities (after the impulse) from the pre-velocities (before the impulse), it is only necessary to determine the remaining Lagrange multiplier $\bar{\mu}$.

We also assume that during the impact the unique impulsive force acting is due to the restitution of the wall determined by $\Psi(t,q) \geqslant 0$.

By applying the techniques of the time-independent case, we construct the following time-dependent projectors along the points of N_t:

$$\tilde{\mathcal{Q}}_t : TQ \longrightarrow \mathcal{P}(T^\perp N_t)$$

$$\tilde{\mathcal{P}}_t : TQ \longrightarrow (\mathcal{P}(T^\perp N_t))^\perp$$

for all $t \in \mathbb{R}$. Therefore, we have $\tilde{\mathcal{P}}_t(\dot{c}(t^+)) = \tilde{\mathcal{P}}_t(\dot{c}(t^-))$.

Let α be the restitution coefficient of the wall. The assumption now is that

$$\dot{c}(t^+)^\perp - V^\perp_{(t,c(t))} = -\alpha(\dot{c}(t^-)^\perp - V^\perp_{(t,c(t))}) \tag{14}$$

where $V_{(t,c(t))}$ is the instantaneous velocity of the wall. That is,

$$\mathrm{d}\Psi_t\,(\dot{c}(t^+)) - \mathrm{d}\Psi_t\,(V_{(t,c(t))}) = -\alpha(\mathrm{d}\Psi_t\,(\dot{c}(t^-)) - \mathrm{d}\Psi_t\,(V_{(t,c(t))})). \tag{15}$$

If $\mathcal{C}_t = g(\mathcal{P}(\operatorname{grad}\Psi_t), \mathcal{P}(\operatorname{grad}\Psi_t))$, $W_{(t,c(t))} = \mathcal{C}_t^{-1}[\mathrm{d}\Psi_t\,(V_{(t,c(t))})]\mathcal{P}(\operatorname{grad}\Psi_t)(c(t))$ and β^0_t is the 1-form given by $\beta^0_t(X) = g(\mathcal{P}(\operatorname{grad}\Psi_t), X)$, for all X, then we can rewrite equation (15) as

$$\beta^0_t(\dot{c}(t^+)) = -\alpha\,\beta^0_t(\dot{c}(t^-)) + (1+\alpha)\,\beta^0_t(W_{(t,c(t))}) \tag{16}$$

or, equivalently,

$$\tilde{\mathcal{Q}}_t\left(\dot{c}(t^+)\right) - \tilde{\mathcal{Q}}_t\left(W_{(t,c(t))}\right) = -\alpha\left(\tilde{\mathcal{Q}}_t\left(\dot{c}(t^-)\right) - \tilde{\mathcal{Q}}_t\left(W_{(t,c(t))}\right)\right).$$

Note that if $V_{(t,c(t))} \in \mathcal{D}_{c(t)}$ then

$$\beta^0_t(V_{(t,c(t))}) = \beta^0_t(W_{(t,c(t))})$$

and, in this case, we could use $V_{(t,c(t))}$ instead of $W_{(t,c(t))}$ in all the computations.

From the above equations, we have

$$\dot{c}(t^+) = (\tilde{\mathcal{P}}_t - \alpha\tilde{\mathcal{Q}}_t)\dot{c}(t^-) + (1+\alpha)\tilde{\mathcal{Q}}_t(W_{(t,c(t))}). \tag{17}$$

Observe that if we consider the velocity of 'approach' $\dot{c}_W(t^-) = \dot{c}(t^-) - W_{(t,c(t))}$, and the velocity of 'separation' $\dot{c}_W(t^+) = \dot{c}(t^+) - W_{(t,c(t))}$ then equation (17) can be rewritten as

$$\dot{c}_W(t^+) = (\tilde{\mathcal{P}}_t - \alpha\tilde{\mathcal{Q}}_t)(\dot{c}_W(t^-)). \tag{18}$$

Theorem 4.1 (Carnot's theorem). *Let*

$$T_1' = \tfrac{1}{2}g(\dot{c}(t^+) - W_{(t,c(t))}, \dot{c}(t^+) - W_{(t,c(t))})$$

be the kinetic energy of the velocity of 'separation',

$$T_0' = \tfrac{1}{2}g(\dot{c}(t^-) - W_{(t,c(t))}, \dot{c}(t^-) - W_{(t,c(t))})$$

be the kinetic energy of the velocity of 'approach', and

$$T_l = \tfrac{1}{2}g(\dot{c}(t^+) - \dot{c}(t^-), \dot{c}(t^+) - \dot{c}(t^-))$$

be the kinetic energy due to the 'loss of velocity'. Then, we have

$$T_1' - T_0' = -\frac{1-\alpha}{1+\alpha}T_l.$$

Proof. It follows from lemma 3.1 and equation (18). □

4.2. Mechanical systems subjected to a time-dependent holonomic one-sided constraint and non-holonomic instantaneous constraints. Carnot's theorem

As in subsection 3.2 we will assume that non-holonomic linear constraints on the velocities are imposed instantaneously, and that the wall is determined by a time-dependent holonomic constraint $\Psi(t, q) \geqslant 0$. If the non-holonomic constraints are given by

$$\psi^A(t, q, \dot{q}) = \nu_i^A(t, q)\dot{q}^i \qquad 1 \leqslant A \leqslant l$$

we can consider for any (t, q) such that $\Psi(t, q) = 0$ the vector space $(F_t)(q)$ annihilated by

$$\langle(\nu_t^A = (\nu_i^A)_t \, \mathrm{d}q^i)_{|q}\rangle \qquad 1 \leqslant A \leqslant l.$$

The equations of motion for this mechanical system are:

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\operatorname{grad} V(c(t)) + \lambda(\dot{c}(t)) \quad \text{and} \quad \dot{c}(t) \in \mathcal{D}(c(t)) \qquad \text{if} \quad \Psi(t, c(t)) > 0$$

$$\Delta\dot{c}(t) = \dot{c}(t^+) - \dot{c}(t^-) \in \mathcal{D}^\perp(c(t)) + T_{c(t)}^\perp N_t + F_t^\perp(c(t)) \qquad \text{if} \quad \Psi(t, c(t)) = 0.$$

We have again that

$$\Delta\dot{c}(t) \in \mathcal{P}(T_{c(t)}^\perp N_t + F_t^\perp(c(t))).$$

In order to determine completely the velocities after the impulse from the velocities before the impulse it is necessary to assume additional conditions, for instance, we will suppose that the restitution coefficient of the wall is $\alpha \in [0, 1]$ and that the velocity after the impact is annihilated by the non-holonomic instantaneous constraints.

Define the following time-dependent complementary orthogonal projectors:

$$\mathbb{Q}_t : TQ \longrightarrow \mathcal{P}(T^\perp N_t + F_t^\perp)$$

$$\mathbb{P}_t : TQ \longrightarrow (\mathcal{P}(T^\perp N_t + F_t^\perp))^\perp$$

along the points of N_t. For each $t \in \mathbb{R}$, denote by Y_t^A, $0 \leqslant A \leqslant l$, the vector fields along the points of N_t defined by

$$g(X, Y_t^0) = X(\Psi_t) \qquad g(X, Y_t^A) = \nu_t^A(X)$$

for all X and $1 \leqslant A \leqslant l$. Then, the projector $\mathbb{Q}_t$ is given by

$$\mathbb{Q}_t = \sum_{0\leqslant A,B\leqslant l} (\mathcal{C}_{AB})_t\, \mathcal{P}(Y_t^A) \otimes \beta_t^B$$

where $\beta_t^B(X) = g(\mathcal{P}(Y_t^B), X)$ for any vector field X, and $(\mathcal{C}_{AB})_t$ denote the entries of the inverse matrix of the matrix whose entries are $\mathcal{C}_t^{AB} = g(\mathcal{P}(Y_t^A), \mathcal{P}(Y_t^B))$.

We assume again that (14) holds, that is, we have

$$\dot c(t^+)^\perp - V^\perp_{(t,c(t))} = -\alpha(\dot c(t^-)^\perp - V^\perp_{(t,c(t))})$$

where α is the restitution coefficient of the wall. Then,

$$\mathrm{d}\Psi_t\,(\dot c(t^+)) - \mathrm{d}\Psi_t\,(V_{(t,c(t))}) = -\alpha(\mathrm{d}\Psi_t\,(\dot c(t^-)) - \mathrm{d}\Psi_t\,(V_{(t,c(t))})). \tag{19}$$

In addition, we have the supplementary conditions

$$\nu_t^A(\dot c(t^+) - V_{(t,c(t))}) = 0 \qquad 1 \leqslant A \leqslant l. \tag{20}$$

Denote

$$W_{(t,c(t))} = \sum_{0\leqslant A,B\leqslant l} (\mathcal{C}_{AB})_t \left[g(Y_t^A, V_{(t,c(t))})\right] \mathcal{P}(Y_t^B).$$

Then,

$$\beta_t^A(W_{(t,c(t))}) = g(Y_t^A, V_{(t,c(t))}) \qquad 0 \leqslant A \leqslant l$$

or, in other words,

$$\mathrm{d}\Psi_t\,(V_{(t,c(t))}) = \beta_t^0(W_{(t,c(t))}) \qquad \textit{and} \qquad \nu_t^A(V_{(t,c(t))}) = \beta_t^A(W_{(t,c(t))}) \qquad 1 \leqslant A \leqslant l.$$

Thus, since $\dot c(t^+) \in \mathcal{D}(c(t))$ and $\dot c(t^-) \in \mathcal{D}(c(t))$, we can rewrite equation (19) as

$$\beta_t^0(\dot c(t^+)) = -\alpha\, \beta_t^0(\dot c(t^-)) + (1+\alpha)\, \beta_t^0(W_{(t,c(t))}). \tag{21}$$

Also equation (20) is rewritten as

$$\beta_t^A(\dot c(t^+) - W_{(t,c(t))}) = 0 \qquad 1 \leqslant A \leqslant l. \tag{22}$$

Proposition 4.2. *Assuming the above conditions, we obtain that*

$$\dot c_W(t^+) = (\mathbb{P}_t - \alpha\tilde{\mathcal{T}}_t)\dot c_W(t^-) \tag{23}$$

where

$$\tilde{\mathcal{T}}_t = \sum_{0\leqslant A\leqslant l} (\mathcal{C}_{A0})_t\, \mathcal{P}(Y_t^A) \otimes \beta_t^0$$

$\dot c_W(t^+) = \dot c(t^+) - W_{(t,c(t))}$ *and* $\dot c_W(t^-) = \dot c(t^-) - W_{(t,c(t))}$.

Proof. Proceeding as in the proof of proposition 3.4 we deduce that

$$\dot{c}(t^+) = \mathbb{P}_t(\dot{c}(t^-)) + \sum_{0\leqslant A,B\leqslant l} (\mathcal{C}_{AB})_t \beta_t^B(\dot{c}(t^+))\mathcal{P}(Y_t^A).$$

Thus, using (21) and (22) it follows that

$$\begin{aligned}\dot{c}(t^+) &= \mathbb{P}_t(\dot{c}(t^-)) - \alpha\tilde{\mathcal{T}}_t(\dot{c}(t^-)) + \alpha\tilde{\mathcal{T}}_t(W_{(t,c(t))}) + \mathbb{Q}_t(W_{(t,c(t))})\\ &= (\mathbb{P}_t - \alpha\tilde{\mathcal{T}}_t)(\dot{c}_W(t^-)) + W_{(t,c(t))}\end{aligned}$$

from which we obtain (23). □

Theorem 4.3 (Extended Carnot's theorem). *Let*

$$T_1' = \tfrac{1}{2}g(\dot{c}(t^+) - W_{(t,c(t))}, \dot{c}(t^+) - W_{(t,c(t))})$$

be the kinetic energy of the velocity of 'separation'

$$T_0' = \tfrac{1}{2}g(\dot{c}(t^-) - W_{(t,c(t))}, \dot{c}(t^-) - W_{(t,c(t))})$$

be the kinetic energy of the velocity of 'approach'

$$T_l = \tfrac{1}{2}g(\dot{c}(t^+) - \dot{c}(t^-), \dot{c}(t^+) - \dot{c}(t^-))$$

be the kinetic energy due to the 'loss of velocity' and $\boldsymbol{C}_t = \mathrm{id} - \mathbb{P}_t - \tilde{\mathcal{T}}_t$. *Then*

$$T_1' - T_0' = -\frac{1-\alpha}{1+\alpha}T_l - \frac{2\alpha}{1+\alpha}T'_{C_t^-} - \alpha g(\tilde{\mathcal{T}}_t(\dot{c}_W(t^-)), \boldsymbol{C}_t(\dot{c}_W(t^-)))$$

where $T'_{C_t^-}$ *is the kinetic energy of the velocity* $\boldsymbol{C}_t(\dot{c}_W(t^-))$.

Moreover, if $\mathcal{P}(Y_t^0) \in F_t$, *we have*

$$T_1' - T_0' = -\frac{1-\alpha}{1+\alpha}T_l - \frac{2\alpha}{1+\alpha}T'_{C_t^-}.$$

Proof. Similar to that of theorems 3.9 and 3.11. □

Example 4.4. We continue example 3.13 with the new assumption that the sphere hits the moving wall given by $x = f(t)$ (we assume thus that the trajectory of any individual particle of the wall is perpendicular to the plane $x = 0$).

The kinetic energy and the permanent constraints are the same as in example 3.13, and apart from the time-dependent holonomic one-sided constraint $\Psi = x - f(t)$ we have, as in example 3.13, the instantaneous constraints ψ^1 and ψ^2 given by (12).

For any fixed time t, we have $\mathbb{Q}_t = \mathbb{Q}$ and $\tilde{\mathcal{T}}_t = \tilde{\mathcal{T}}$, where $\mathbb{Q}$ and $\tilde{\mathcal{T}}$ are as in example 3.13.

The velocity of the wall is

$$V_{(t,c(t))} = (\dot{f}(t), 0, 0, 0, 0, 0)$$

and

$$W_{(t,c(t))} = (\dot{f}(t), 0, 0, 0, \dot{f}(t)/r, 0).$$

Note that $W_{(t,c(t))} \in F_t$, for all t.

Consider now a pre-impact velocity which verifies the permanent non-holonomic constraints, that is,

$$(\dot{x}_0, \dot{y}_0, 0, -\dot{y}_0/r, \dot{x}_0/r, (\omega_z)_0).$$

If we express $\mathbb{P}_t - \alpha\tilde{\mathcal{T}}_t$ by means of a 6×6-matrix

$$\mathcal{M} = \begin{pmatrix} 1-\frac{r^2+\alpha r^2}{k^2+r^2} & 0 & 0 & 0 & -\frac{k^2 r(1+\alpha)}{k^2+r^2} & 0 \\ 0 & 1-\frac{k^2r^2}{(k^2+r^2)(2k^2+r^2)} & 0 & \frac{k^4 r}{(k^2+r^2)(2k^2+r^2)} & 0 & \frac{k^2 r}{2k^2+r^2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{k^2 r}{(k^2+r^2)(2k^2+r^2)} & 0 & 1-\frac{k^4}{(k^2+r^2)(2k^2+r^2)} & 0 & -\frac{k^2}{2k^2+r^2} \\ -\frac{r(\alpha+1)}{k^2+r^2} & 0 & 0 & 0 & 1-\frac{k^2(\alpha+1)}{k^2+r^2} & 0 \\ 0 & \frac{r}{2k^2+r^2} & 0 & -\frac{k^2}{2k^2+r^2} & 0 & 1-\frac{k^2+r^2}{2k^2+r^2} \end{pmatrix}$$

then the relationship between the velocity of 'separation' and the velocity of 'approach' is given by

$$\begin{aligned}&\left(\dot{x}_1 - \dot{f}(t), \dot{y}_1, \dot{z}_1, (\omega_x)_1, (\omega_y)_1 - \dot{f}(t)/r, (\omega_z)_1\right) \\ &\qquad = \left(\dot{x}_0 - \dot{f}(t), \dot{y}_0, 0, -\frac{\dot{y}_0}{r}, \frac{\dot{x}_0 - \dot{f}(t)}{r}, (\omega_z)_0\right)\mathcal{M}.\end{aligned}$$

As in example 3.13, for initial velocities satisfying $\dot{y}_0 = r(\omega_z)_0$, we have that the relation between the kinetic energies of the 'separation' and 'approach' velocities are

$$T_1' - T_0' = -\frac{1-\alpha}{1+\alpha}T_l$$

and, if $\dot{y}_0 \neq r(\omega_z)_0$, then

$$T_1' - T_0' = -\frac{(1-\alpha)}{1+\alpha}T_l - \frac{2\alpha}{\alpha+1}T'_{C_t^-}$$

where

$$T'_{C_t^-} = \frac{k^2(k^2+r^2)}{2r^2(2k^2+r^2)}(\dot{y}_0 - r(\omega_z)_0)^2.$$

Acknowledgments

This work has been partially supported through grants DGICYT (Spain) (projects PB97-1257, PB97-1487 and BMF2000-0808), CONACYT (Mexico) (projects 400200-5-1406PE and 32167-E) and NATO collaborative research grant (no CRG 950833). ML and DMD wish to express their gratitude for the warm hospitality offered to them in the Departamento de Matemática Fundamental (University of La Laguna) where part of this work was done. EAL acknowledges the hospitality of the Departamento de Economía Aplicada (Matemáticas), University of Valladolid (Spain). Finally, EAL and PP acknowledge IMAFF (CSIC, Spain) for hospitality when part of this paper was prepared. We also thank the referees for various useful comments.

References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (Reading, MA: Benjamin-Cummings)
[2] Appell P 1953 *Traité de Mécanique Rationelle* vol II (Paris: Gauthiers-Villars)
[3] Bloch A M, Krishnaprasad P S, Marsden J E and Murray R M 1996 Nonholonomic mechanical systems with symmetry *Arch. Rational Mech. Anal.* **136** 21–99
[4] Brogliato B 1996 *Nonsmooth Impact Mechanics. Models, Dynamics and Control (Lecture Notes in Control and Information Science vol 220)* (New York: Springer)

[5] Chetaev N G 1989 *Theoretical Mechanics* (Moscow: Mir, Berlin: Springer)

[6] Cushman R and Śniatycki S 1998 (ed) *Proc. Pacific Institute of Mathematical Sciences. Workshop on Nonholonomic Constraints in Dynamics (Calgary, 1997) Rep. Math. Phys.* **42** no 1, 2

[7] Fusco G and Oliva M 1986 Dissipative systems with constraints *J. Diff. Eq.* **63** 362–88

[8] Ibort A, de León M, Lacomba E A, Martín de Diego D and Pitanga P 1997 Mechanical systems subjected to impulsive constraints *J. Phys. A: Math. Gen.* **30** 5835–54

[9] Ibort A, de León M, Lacomba E A, Marrero J C, Martín de Diego D and Pitanga P 1998 Geometric formulation of mechanical systems subjected to time-dependent one-sided constraints *J. Phys. A: Math. Gen.* **31** 2655–74

[10] Kane T R 1968 *Dynamics* (New York: Holt, Rinehart and Winston)

[11] Koiller J 1992 Reduction of some classical non-holonomic systems with symmetry *Arc. Rat. Mech. Anal.* **118** 113–48

[12] Lacomba E A and Tulczyjew W A 1990 Geometric formulation of mechanical systems with one-sided constraints *J. Phys. A: Math. Gen.* **23** 2801–13

[13] Lewis A D 1998 Affine connections and distributions with applications to nonholonomic mechanics *Rep. Math. Phys.* **42** 135–64

[14] Mladenova C 1995 Dynamics of nonholonomic systems *Z. angew. Math. Mech.* **75** 199–205

[15] Neimark J I and Fufaev N A 1972 *Dynamics of Nonholonomic Systems (Trans. Am. Math. Soc.)* (Providence, RI: American Mathematical Society)

[16] Painlevé P 1930 *Cours de Mécanique* vol I (Paris: Gauthiers-Villars)

[17] Pars L A 1965 *A Treatise on Analytical Dynamics* (Woodbridge, CT: Ox Bow) (reprinted in 1979)

[18] Rosenberg R M 1977 *Analytical Dynamics of Discrete Systems* (New York: Plenum)

[19] Stewart D 2000 Rigid-body dynamics with friction and impact *SIAM Rev.* **42** 3–39

[20] Tulczyjew W M 1984 *Geometrical Formulation of Physical Theories: Statics and Dynamics of Mechanical Systems* (Naples: Bibliopolis)

[21] Whittaker E T 1988 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)